JOURNAL OF APPROXIMATION THEORY 39, 241–246 (1983)

Some Remarks on a Theorem of Saff and Varga

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Communicated by Richard S. Varga

Received May 11, 1982

Let $R[D_\rho]$ be the collection of functions analytic on the disk $D_\rho = \{z \mid |z| < \rho\}$, and let A_ρ be the set of all functions from $R[D_\rho]$ which have at least one singularity on the circle $|z| = \rho$. Further, let $\bar{D}_\rho = \{z \mid |z| \leqslant \rho\}$.

Let us denote by $p_{n-1}(f, z)$ the Lagrange interpolatory polynomial of $f \in A_\rho$ on the nth roots of unity, and if $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then let

$$P_{n-1,j}(f, z) = \sum_{k=0}^{n-1} a_{k+jn} z^k,$$

$$d_{n-1,l}(f, z) = p_{n-1}(f, z) - \sum_{j=0}^{l-1} P_{n-1,j}(f, z).$$

Then, a result of Cavaretta *et al.* [1] can be stated as

THEOREM A. *For each $f \in A_\rho$ $(\rho > 1)$ and each positive integer l,*

$$\lim_{n \to \infty} d_{n-1,l}(f, z) = 0 \qquad \text{if} \quad z \in D_{\rho^{l+1}}, \tag{1}$$

the convergence being uniform and geometric on any closed subset of $D_{\rho^{l+1}}$. Moreover, (1) is best possible in the sense that there is some $\tilde{f} \in A_\rho$ and some $\tilde{z}$ with $|\tilde{z}| = \rho^{l+1}$ for which $d_{n-1,l}(\tilde{f}, \tilde{z})$ does not tend to zero as $n \to \infty$.

The case $l = 1$ was proved by Walsh [2]. In [3], Saff and Varga investigated $d_{n-1,l}(f, z)$ if $|z| > \rho^{l+1}$. They established

THEOREM B. *For each $f \in A_\rho$ and for each positive integer l, $d_{n-1,l}(f, z)$ can be bounded in at most l distinct points in $|z| > \rho^{l+1}$. This result is sharp.*

0021-9045/83 $3.00

in the sense that, given any l distinct points $\{\eta_k\}_{k=1}^{l}$ in the annulus $\rho^{l+1} < |z| < \rho^{l+2}$, there is an $\hat{f} \in A_\rho$ for which

$$\lim_{n\to\infty} d_{n-1,l}(\hat{f}, \eta_k) = 0, \qquad k = 1, 2, \dots, l. \tag{2}$$

Concerning to the latter result, the following questions arise:

(i) In Theorem B, is the restriction $|\eta_k| < \rho^{l+2}$ necessary?

(ii) For which functions is (2) true with prescribed $\{\eta_k\}_{k=1}^{l}$?

(iii) What is the situation in the case $\rho = 1$?

In the following, we deal with these questions. Let s, l and L be positive integers with $s \leqslant l < L$, let $\rho > 1$, and let $\{\eta_k\}_{k=1}^{s}$ be distinct points with $\rho^{l+1} < |\eta_k| < \rho^{L+1}$ $(k = 1, 2, \dots, s)$. Further, let $\varphi \in A_{\rho^r}$ and $\psi \in R[\bar{D}_{\alpha_s^{1/(l+1)}}]$, where r is the least common multiple of $\{l+1, l+2, \dots, L\}$ and $\alpha_s = \max_{1 \leqslant k \leqslant s} |\eta_k|$.

THEOREM 1. *Let $\omega_s(z) = \prod_{k=1}^{s}(z - \eta_k)$ and let*

$$f(z) = \omega_s(z)\,\varphi(z^r) + \psi(z). \tag{3}$$

Then, $f \in A_\rho$ and

$$\lim_{n\to\infty} d_{n-1,l}(f, \eta_k) = 0 \qquad (k = 1, 2, \dots, s). \tag{4}$$

Proof. As $\alpha_s^{1/(l+1)} > \rho$, then ψ is analytic in $\bar{D}_\rho$, and as $\varphi(z^r) \in A_\rho$, it follows that $f \in A_\rho$.

Now let $f \in A_\rho$ be arbitrary, and write $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Further, let $g(z) = f(z)/\omega_s(z) = \sum_{k=0}^{\infty} b_k z^k$. It follows that $g \in A_\rho$ and $a_k = \sum_{j=0}^{s} \beta_j b_{k-j}$ if $k \geqslant s$, where $\omega_s(z) = \sum_{j=0}^{s} \beta_{j,s} z^j = \sum_{j=0}^{s} \beta_j z^j$. Since $p_{n-1}(f, z) = \sum_{k=0}^{n-1} z^k \sum_{j=0}^{\infty} a_{k+jn}$ (see, e.g., [1, p. 160, Eq. (2.9)]),

$$d_{n-1,l}(f, z) = \sum_{k=0}^{n-1} z^k \sum_{j=l}^{\infty} a_{k+jn} = \sum_{k=0}^{n-1} \sum_{j=l}^{\infty} \sum_{i=0}^{s} z^k \beta_i b_{jn+k-i} = \sum_{j=l}^{\infty} \tau(l).$$

Here

$$\tau(l) = \sum_{i=0}^{s} \sum_{k=0}^{n-1} (\beta_i z^i)(b_{jn+k-i} z^{k-i}) = \sum_{i=0}^{s} (\beta_i z^i) \left\{ \sum_{0 \leqslant k < i} + \sum_{k=i}^{n+i-1} - \sum_{n \leqslant k < n+i} \right\}$$

$$= \omega_s(z) \sum_{k=0}^{n-1} b_{jn+k} z^k + \sum_{i=1}^{s} \beta_i \sum_{k=0}^{i-1} b_{jn+k-i} z^k$$

$$- z^n \sum_{i=1}^{s} \beta_i \sum_{k=0}^{i-1} b_{n(j+1)+k-i} z^k,$$

so

$$d_{n-1,l}(f,z) = \omega_s(z)\, d_{n-1,l}(g,z) + \sum_{i=1}^{s} \beta_i \sum_{k=0}^{i-1} z^k \sum_{j=l}^{\infty} b_{jn+k-i}$$
$$- z^n \sum_{i=1}^{s} \beta_i \sum_{k=0}^{i-1} z^k \sum_{j=l+1}^{\infty} b_{jn-k-i}.$$

Hence, denoting

$$h_n(z) = \sum_{i=1}^{s} \beta_i \sum_{k=0}^{i-1} z^k \sum_{j=l+1}^{\infty} b_{jn+k-i},$$

(4) is true iff

$$\lim_{n\to\infty} \eta_k^n h_n(\eta_k) = 0 \qquad (k = 1, 2,..., s). \tag{5}$$

We next write $h_n(z)$ as

$$h_n(z) = \sum_{i=1}^{s} \beta_i \sum_{k=0}^{i-1} z^k \sum_{j=l+1}^{L} b_{jn+k-i} + \sum_{i=1}^{s} \beta_i \sum_{k=0}^{i-1} z^k \sum_{j=L+1}^{\infty} b_{jn+k-i}$$
$$= h_{n,1}(z) + h_{n,2}(z).$$

Let $\varepsilon > 0$ be such that $\alpha_s < (\rho - \varepsilon)^{L+1}$. Since $g \in A_\rho$, then $\overline{\lim}_{k\to\infty} |b_k|^{1/k} = 1/\rho$, hence $|b_k| < 1/(\rho - \varepsilon)^k$ if $k > k_0(\varepsilon)$. Thus,

$$|\eta_k^n h_{n,2}(\eta_k)| = 0\left(\frac{|\eta_k|^n}{(\rho-\varepsilon)^{n(L+1)}}\right) = 0\left(\left(\frac{\alpha_s}{(\rho-\varepsilon)^{L+1}}\right)^n\right) \to 0 \qquad \text{as} \quad n \to \infty.$$

Now we show that if $g(z) = \varphi(z^r)$, then $h_{n,1}(z) = 0$. If r is not a divisor of m, then $b_m = 0$. If $l + 1 \leqslant j \leqslant L$, then $j \mid r$ by the definition of r. If $0 \leqslant k < i \leqslant s \leqslant l$, then j is not a divisor of $i - k$ so r is not a divisor of $jn + k - i$. Hence $b_{jn+k-i} = 0$ and so $h_{n,1}(z) = 0$. From this (5) follows when $f(z) = \omega_s(z)\, \varphi(z^r)$ so in this case (4) is true. It follows from Theorem A that if $f = \psi$, then (4) is true, but as $d_{n-1,l}$ is a linear operator, we have proved our statement. Q.E.D.

In Theorem 1 we gave a sufficient condition for (4). In the case of $s = l$ and $L = l + 1$ we give a necessary condition for (2).

THEOREM 2. *If $f \in A_\rho$, $\rho^{l+1} < |\eta_k| < \rho^{l+2}$ $(k = 1, 2,..., l)$ and (4) fulfills with $s = l$ then*

$$f(z) = \omega_l(z)\, \varphi(z^{l+1}) + \psi(z), \tag{6}$$

where $\varphi \in A_{\rho^{l+1}}$ and $\psi \in R[D_{\gamma_l^{1/(l+1)}}]$. Here $\gamma_l = \min_{1 \leqslant k \leqslant l} |\eta_k|$.

Proof. From the proof of Theorem 1 it is obvious that—using the same notations—

$$\lim_{n\to\infty} \eta_k^n h_{n,1}(\eta_k)=0 \qquad (k=1,2,\dots,l). \tag{7}$$

By the definition of $h_{n,1}$, we may write

$$h_{n,1}(\eta_k)=\sum_{i=1}^{l}\beta_i\sum_{j=0}^{i-1}\eta_k^j b_{n(l+1)+j-i}=\sum_{j=1}^{l} b_{n(l+1)-j}\sum_{i=j}^{l}\beta_i\eta_k^{i-j}. \tag{8}$$

First we prove that the system of equations

$$\sum_{j=1}^{l}\alpha_l(j)\sum_{i=j}^{l}\beta_{i,l}\eta_k^{i-j}=\xi_{k,l} \qquad (k=1,\dots,l). \tag{9}$$

is solvable for the $\alpha_l(j)$'s as "unknowns," with arbitrary $\xi_{k,l}$. We prove it by induction on l. If $l=2$, then it follows by easy computation.

Let $\omega_{l+1}(z)=\omega_l(z)(z-\eta_{l+1})=\sum_{i=0}^{l+1}\beta_{i,l+1}z^i$. Then

$$\begin{aligned}\beta_{i,l+1}&=-\beta_{0,l}\eta_{l+1} && \text{if}\quad i=0,\\ &=\beta_{i-1,l}-\eta_{l+1}\cdot\beta_{i,l} && \text{if}\quad 0<i<l+1,\\ &=1 && \text{if}\quad i=l+1.\end{aligned}$$

$$\begin{aligned}\xi_{k,l+1}&=\sum_{j=1}^{l+1}\alpha_{l+1}(j)\sum_{i=j}^{l+1}\beta_{i,l+1}\eta_k^{i-j}\\
&=\sum_{j=1}^{l}\alpha_{l+1}(j)\left\{\sum_{i=j}^{l}(\beta_{i-1,l}-\eta_{l+1}\beta_{i,l})\,\eta_k^{i-j}+\beta_{l,l}\eta_k^{l+1-j}\right\}\\
&\quad+\alpha_{l+1}(l+1)\\
&=\sum_{j=1}^{l}\alpha_{l+1}(j)\left\{\beta_{j-1,l}+(\eta_k-\eta_{l+1})\sum_{i=j}^{l}\beta_{i,l}\eta_k^{i-j}\right\}+\alpha_{l+1}(l+1)\\
&=\sum_{j=1}^{l}\alpha_{l+1}(j)\beta_{j-1,l}+(\eta_k-\eta_{l+1})\sum_{j=1}^{l}\alpha_{l+1}(j)\sum_{i=j}^{l}\beta_{i,l}\eta_k^{i-j}\\
&\qquad\qquad (k=1,2,\dots,l+1).\end{aligned}$$

So

$$\frac{\xi_{k,l+1}-\xi_{l+1,l+1}}{\eta_k-\eta_{l+1}}=\sum_{j=1}^{l}\alpha_{l+1}(j)\sum_{i=j}^{l}\beta_{i,l}\eta_k^{i-j} \qquad (k=1,2,\dots,l).$$

Hence we have reduced the case $l+1$ to the case l and completed the inductive proof.

Writing, in (9), $b_{n(l+1)-j}$ instead of $\alpha_l(j)$, we get, by (7), $|\xi_{k,l}| = o(|\eta_k|^{-n})$ ($k = 1, 2,..., l$). Solving (9) for $b_{n(l+1)-j}$, we obtain

$$|b_{n(l+1)-j}| = o(\max |\eta_k|^{-n}) = o(\gamma_l^{-n}) \qquad (j = 1, 2,..., l), \tag{10}$$

i.e., $\overline{\lim}_{n\to\infty} |b_{n(l+1)-j}|^{1/(n(l+1)-j)} \leqslant \gamma_l^{-1/(l+1)}$ ($j = 1, 2,..., l$). Thus,

$$g(z) = \sum_{j=0}^{\infty} b_{j(l+1)} z^{j(l+1)} + \sum_{n=0}^{\infty} \sum_{j=1}^{l} b_{n(l+1)+j} z^{n(l+1)-j} = g_1(z) + g_2(z).$$

From (10), $g_2 \in R[D_{\gamma_l^{1/(l+1)}}]$. Now, $g \in A_\rho$ so that $g_1 \in A_\rho$ as well; hence $g_1(z) = \varphi(z^{l+1})$, where $\varphi \in A_{\rho^{l+1}}$ which gives the desired result. Q.E.D.

Remark 1. There exists a $g \in A_1$ such that, for any $\eta \in \mathbb{C}$,

$$\lim_{n\to\infty} d_{n-1,l}(f_\rho, \eta) = 0 \qquad (\rho \geqslant 1),$$

where $f_\rho(z) = (z-\eta)\, g(z/\rho)$ (here g is independent of the choice of η).

Proof. Let $g(z) = \sum_p p^{-2} z^{p-1}$, where p runs over the primes. Evidently $g \in A_1$ and g is continuous in $\bar{D}_1$; so $d_{n-1,l}(f_\rho)$ exists even in the case $\rho = 1$. In the case $s = 1$, $h_n(z)$ takes the form

$$h_n(z) = \sum_{j=l+1}^{\infty} b_{jn-1},$$

so $h_n(z)$ is identically 0 because a comosite number minus one is never equal to a prime number minus one. Hence, by (5), we have proved our statement.

Remark 2. Above we gave an example of a function $f \in A_1$ for which $d_{n-1,l}(f, z)$ converges to 0 at a point outside $|z| \leqslant 1$. It remains an open question whether a function $f \in A_1$ exists such that $d_{n-1,l}(f, z)$ converges to zero at several points outside $|z| \leqslant 1$.

Remark 3. Theorem 2 is a partial converse of Theorem 1. It would be interesting to characterize *all* functions for which (4) is true.

Remark 4. In [1], Cavaretta *et al.* proved that if $f \in A_\rho$ ($\rho > 1$) and if f is continuous on $|z| \leqslant \rho$, then

$$\max_{|z| \leqslant \rho^{l+1}} |d_{n-1,l}(f, z)| = O(E_{n-1}(f)), \qquad \text{as} \quad n \to \infty, \tag{11}$$

where

$$E_n(f) = \min \max_{|z| = \rho} |f(z) - p_n(z)|,$$

the minimum being taken over all polynomials $p_n(z)$ of degree $\leqslant n$. The following simple example shows that this estimation is sharp. Let $f \in A_\rho$: $f(z) = \sum_{k=0}^{\infty} (z/\rho)^k b_k$, where $b_k \geqslant 0$. Then

$$\begin{aligned} E_{n-1}(f) &\leqslant \max_{|z| \leqslant \rho} \left| f(z) - \sum_{k=0}^{n-1} (z/\rho)^k b_k \right| \\ &= \max_{|z| \leqslant \rho} \left| \sum_{k=n}^{\infty} (z/\rho)^k b_k \right| = \sum_{k=n}^{\infty} b_k \end{aligned} \tag{12}$$

and

$$\begin{aligned} \max_{|z| \leqslant \rho^{l+1}} |d_{n-1,l}(f, z)| &= \sum_{k=0}^{n-1} \rho^{k(l+1)} \sum_{j=l}^{\infty} \rho^{-k-jn} b_{k+jn} \geqslant \sum_{k=0}^{n-1} \rho^{l(k-n)} b_{k+ln} \\ &\geqslant \rho^{-l} b_{(l+1)n-1}. \end{aligned} \tag{13}$$

Now let $g_l(z) = \sum_{k=0}^{\infty} (z/\rho)^{(l+1)^k - 1}/(l+1)^k$. Evidently $g_l \in A_\rho$ and, by (12), (13) and an easy computation, for all positive integers n and l,

$$\max_{|z| \leqslant \rho^{l+1}} |d_{(l+1)^n-1,l}(g_l, z)| \geqslant \rho^{-l} l/(l+1)^2 \, E_{(l+1)^n-1}(g_l),$$

i.e.,

$$\overline{\lim_{n \to \infty}} \frac{\max_{|z| \leqslant \rho^{l+1}} |d_{n,l}(g_l, z)|}{E_n(g_l)} > 0.$$

This proves the sharpness of (11).

References

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